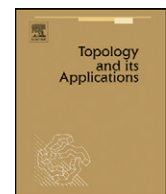




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Actions of totally disconnected groups and equivariant singular homology

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ABSTRACT

We study equivariant singular homology in the case of actions of totally disconnected locally compact groups on topological spaces. Theorem A says that if G is a totally disconnected locally compact group and X is a G -space, then any short exact sequence of covariant coefficient systems for G induces a long exact sequence of corresponding equivariant singular homology groups of the G -space X . In particular we consider the case where G is a totally disconnected compact group, i.e., a profinite group, and G acts freely on X . Of special interest is the case where G is a p -adic group, p a prime. The conjecture that no p -adic group, p a prime, can act effectively on a connected topological manifold, is namely known to be equivalent to the famous Hilbert–Smith conjecture. The Hilbert–Smith conjecture is the statement that, if a locally compact group G acts effectively on a connected topological manifold M , then G is a Lie group.

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0. Introduction

In [4], see also [2] and [3], we introduced equivariant singular homology $H_*^G(-; k)$; where G is any topological group, and k is any given covariant coefficient system for G , over an arbitrary ring R . This equivariant homology theory is defined on the category of all topological G -pairs (X, A) and corresponding G -maps, and it satisfies appropriate equivariant analogues of the Eilenberg–Steenrod axioms.

In particular, the equivariant analogue of the dimension axiom says that $H_n^G(G/H; k) = 0$, if $n \geq 1$, and $H_0^G(G/H; k) \cong k(G/H)$, for every $H < G$. For more information about equivariant singular homology and also for explanations of terminology and notation left undefined, we refer to Section 1. We shall in this paper automatically assume that any given topological space X is Hausdorff.

In this paper we prove that, in the case where G is *totally disconnected and locally compact*, every short exact sequence of covariant coefficient systems induces a corresponding long exact sequence in equivariant singular homology. That is, we prove the following, see Theorem 3.1.

Theorem A. *Suppose G is a totally disconnected locally compact group. Then each short exact sequence $0 \rightarrow k_0 \xrightarrow{\lambda} k_1 \xrightarrow{\mu} k_2 \rightarrow 0$ of covariant coefficient systems for G , over a ring R , induces a long exact sequence in equivariant singular homology, for any G -space X ,*

$$\cdots \xrightarrow{\Delta} H_n^G(X; k_0) \xrightarrow{\lambda_*} H_n^G(X; k_1) \xrightarrow{\mu_*} H_n^G(X; k_2) \xrightarrow{\Delta} H_{n-1}^G(X; k_0) \xrightarrow{\lambda_*} \cdots$$

The result also holds in the relative case, i.e., it holds for a G -pair (X, A) in place of X .

Let us here at once point out that it is not at all clear whether Theorem A holds; for instance, for an arbitrary compact Lie group G . The main technical result behind Theorem A is given in Theorem 2.6 and its Corollary 2.7, which shows that, in

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the case where G is a totally disconnected locally compact group, the equivariant singular chain complex is always a direct sum of coefficient modules. Theorem A is then an immediate consequence of this fact, see Section 3. Using Theorem A it is possible for us to find interesting relations between equivariant singular homology groups (in fact, R -modules) of a G -space X , with different coefficient systems. Thus we should look for suitable coefficient systems for G , depending on the geometric situation, and short exact sequences between these coefficient systems.

In the present paper our interest lies exclusively in actions of totally disconnected locally compact groups G that are *non-discrete*. But let us here anyhow mention, as a side-remark, that for $G = C_p = \{e, g, \dots, g^{p-1}\}$, $g^p = e$, a finite cyclic group of prime order p , and suitable choices of short exact sequences of coefficient systems for C_p , over the field \mathbb{Z}_p , one obtains from Theorem A, for arbitrary C_p -spaces, the well-known exact Smith sequences. We refer to [6] for the details, and for a comparison of different approaches in the literature to the Smith homology sequences. For a concise and elegant treatment of the Smith sequences in the classical case of simplicial actions on abstract simplicial complexes, see Section III.3, in particular, Theorem III.3.3, in [1].

Although not necessary for the actual results the reader is advised to read the present paper as if G always stood for a non-discrete group. We shall, for instance, make no further references to the existing literature, in case where G is assumed discrete.

Let R be an arbitrary but fixed ring, and let G denote any totally disconnected locally compact group. In this paper we investigate the consequences of Theorem A for one specific short exact sequence of coefficient systems for G , over the ring R , namely for the sequence

$$0 \rightarrow I \xrightarrow{\iota} \Lambda \xrightarrow{\mu} \underline{R} \rightarrow 0. \quad (1)$$

Here Λ denotes the *natural coefficient system for the group G , over the ring R* , see Section 4 for the precise definition; here we only note that $\Lambda(G/H) = H_0(G/H; R)$, for $H < G$. By \underline{R} we denote the *(canonical) constant coefficient system for G , over the ring R* , see Section 6; we have that $\underline{R}(G/H) = R$, for every $H < G$. Furthermore $\mu: \Lambda \rightarrow \underline{R}$ denotes the *augmentation homomorphism* which, using the above descriptions of Λ and \underline{R} , equals $p_*: H_0(G/H; R) \rightarrow H_0(G/G; R) \cong R$, for every $H < G$. The *augmentation coefficient system I for G , over the ring R* , is then defined by $I = \text{Ker } \mu$.

In Section 4 we establish the following, see Theorem 4.1.

Theorem B. *For each G -space X , where G is a totally disconnected locally compact group, there exists a natural isomorphism*

$$\Phi: H_n^G(X, \Lambda) \rightarrow H_n(X; R), \quad \text{for every } n \geq 0.$$

The same holds for any G -pair (X, A) in place of X . Here $H_(; R)$ denotes ordinary singular homology, with coefficients in the ring R .*

Combining Theorem B with Theorem A, applied to the short exact sequence (1), we obtain the result below, see Theorem 6.1.

Theorem C. *For each G -space X , where G is a totally disconnected locally compact group, there exists a long exact sequence*

$$\dots \xrightarrow{\Delta} H_n^G(X; I) \xrightarrow{i} H_n(X; R) \xrightarrow{\bar{\mu}} H_n^G(X; \underline{R}) \xrightarrow{\Delta} H_{n-1}^G(X; I) \xrightarrow{i} \dots,$$

and the same holds for any G -pair (X, A) , in place of X .

When G is a totally disconnected *compact* group we have some further results, given in Theorems E and F below. Part of the proof of Theorem E, see Theorem 5.5, is based upon the following lemma, see Lemma 5.4. Lemma D with $G = A_p$; the p -adic group, p a prime; is in [11, Section 3], attributed to C.N. Lee [7].

Lemma D. *Let X be a free G -space, where G is a totally disconnected compact group. Then the natural projection $p: X \rightarrow X/G$ is a fibration.*

The proof that we give of Lemma D uses in a very crucial way the fact that a totally disconnected compact group is a profinite group. Thus we should here recall the following definition and theorem below. Theorem Π is a well-known result, see e.g. Theorem I.2 in [12].

Definition. A profinite group is a topological group G that is the inverse limit of finite groups.

Theorem Π . *A topological group G is a profinite group if and only if G is totally disconnected and compact.*

The other part of the proof of Theorem E uses the result in Theorem 5.3, which in fact holds for G an arbitrary topological group.

Theorem E. Let X be a free G -space, where G is a totally disconnected compact group. Then there exists a natural isomorphism

$$\mathcal{E} : H_n^G(X; \mathbb{R}) \xrightarrow{\cong} H_n(X/G; \mathbb{R}), \quad \text{for each } n \geq 0.$$

The corresponding isomorphism also holds in the relative case.

By using Theorem A, applied to the short exact sequence (1), together with Theorems B and E, we obtain Theorem F below, see Theorem 6.3.

Theorem F. Suppose X is a free G -space, where G is a totally disconnected compact group. Then there exists a long exact sequence

$$\cdots \rightarrow H_n^G(X; I) \xrightarrow{i} H_n(X; \mathbb{R}) \xrightarrow{\pi_*} H_n(X/G; \mathbb{R}) \xrightarrow{\bar{\Delta}} H_{n-1}^G(X; I) \xrightarrow{i} \cdots.$$

Here π_* denotes the homomorphism induced, in ordinary singular homology, by the natural projection $\pi : X \rightarrow X/G$. The corresponding result holds in the relative case.

We conclude this introduction by pointing out that the above results in particular hold for each of the p -adic groups A_p , where p is a prime. Recall that

$$A_p = \varprojlim_r (\mathbb{Z}_p \xleftarrow{\alpha_1} \mathbb{Z}_{p^2} \xleftarrow{\alpha_2} \cdots \leftarrow \mathbb{Z}_{p^r} \xleftarrow{\alpha_r} \mathbb{Z}_{p^{r+1}} \leftarrow \cdots),$$

where p is a prime and $\mathbb{Z}_{p^r} = \mathbb{Z}/p^r\mathbb{Z}$, $r \geq 1$, denotes the group of the integers modulo $p^r\mathbb{Z}$, and $\alpha_r : \mathbb{Z}_{p^{r+1}} \rightarrow \mathbb{Z}_{p^r}$ is the homomorphism for which $\alpha_r([1]_{p^{r+1}}) = [1]_{p^r}$. Thus A_p is, by definition, a profinite group, and hence A_p is a totally disconnected compact group, by Theorem Π .

Our interest in studying actions of the p -adic groups A_p , p a prime, is largely motivated by their connection with the famous Hilbert–Smith conjecture, namely the following long standing conjecture.

Hilbert–Smith conjecture. Suppose G is a locally compact group which acts effectively on a connected topological manifold M . Then G is a Lie group.

It is a well-known fact that the Hilbert–Smith conjecture equivalent to following conjecture.

Conjecture Γ . The p -adic group A_p , where p is a prime, cannot act effectively on a connected topological manifold.

Although the fact that the Hilbert–Smith conjecture is equivalent to Conjecture Γ is well known, it is not easy to find an explicit and correct proof in the published literature. There are published papers that claim to prove this equivalence, but which contain serious mistakes. A correct and explicit proof of the equivalence of the Hilbert–Smith conjecture and Conjecture Γ can be found in [8].

The Hilbert–Smith conjecture is known to hold when $\dim M \leq 2$, see [9, Theorem 6.1.3 on p. 233, and Theorem on p. 249]. It is not known whether a p -adic group A_p , p a prime, can act effectively on the 3-dimensional Euclidean space \mathbb{R}^3 . The Hilbert–Smith conjecture is also known under some additional assumptions on the group G , the manifold M , and the action of G on M , see e.g. the introduction in [11].

The present paper is to be considered as a first one in a series of papers to follow, all of which, in one way or another, are connected with investigations into Conjecture Γ . For the second paper in this series, see [5].

1. Preliminaries

1.1. The construction of equivariant singular homology

Let G be a topological group. In the definition of a topological group G we include the assumption that G is a T_1 -space, i.e., that each point in G is closed. It then follows that each topological group G is a Hausdorff space, see e.g. [10, p. 97, F]. By a G -space X we mean a topological Hausdorff space X together with a (continuous!) action $\Gamma : G \times X \rightarrow X$, $(g, x) \mapsto gx$, of G on X . Suppose X is a G -space. Then we have for each $x \in X$ that the isotropy subgroup G_x of G at x is a closed subgroup of G , i.e., $G_x < G$. By a G -map $f : X \rightarrow Y$, where X and Y are G -spaces, we mean a G -equivariant map from X to Y , i.e., $f(gx) = gf(x)$, for all $g \in G$, and every $x \in X$.

In this section G denotes an arbitrary topological group. Suppose H is a closed subgroup of G , i.e. $H < G$. Then the homogeneous space G/H is Hausdorff, see e.g. [9, Theorem on p. 27], and G acts on G/H by $G \times G/H \rightarrow G/H$, $(g, \bar{g}H) \rightarrow g\bar{g}H$. Let

$$\alpha : G/H \rightarrow G/K$$

be a G -map, where $H, K < G$, and let us denote $\alpha(eH) = g_0K$. Then

$$\alpha(gH) = gg_0K, \quad \text{for every } gH \in G/H.$$

Since α is a G -map, it follows that $G_x \subset C_{\alpha(x)}$, for every $x \in G/H$. Taking $x = eH \in G/H$ we obtain

$$H \subset g_0Kg_0^{-1}.$$

By a ring R we mean a ring with identity element. We are not assuming that the ring R is commutative. All R -modules will be assumed to be unitary.

Definition 1.1. A covariant coefficient system k for G , over a ring R , is a covariant functor k from the category whose objects are G -spaces of the form G/H , where $H < G$, and whose morphisms are G -homotopy classes of G -maps between such G -spaces, to the category of left R -modules.

Suppose k is a covariant coefficient system for G , over a ring R . In [4], see also [2] and [3], we constructed an equivariant homology theory $H_*^G(\cdot; k)$, defined on the category of all topological G -pairs and G -maps, and with values in the category of left R -modules, which satisfies all seven equivariant Eilenberg–Steenrod axioms. The meaning of the seventh axiom, the equivariant dimension axiom, is the following.

If $H < G$, then

$$H_n^G(G/H; k) = 0, \quad \text{for every } n \neq 0, \quad (1)$$

and for $n = 0$ there exists a natural isomorphism

$$\gamma : H_0^G(G/H; k) \xrightarrow{\cong} k(G/H). \quad (2)$$

Thus, if $\alpha : G/H \rightarrow G/K$ is a G -map, where $H, K < G$, then the diagram

$$\begin{array}{ccc} H_0^G(G/H; k) & \xrightarrow[\cong]{\gamma} & k(G/H) \\ \alpha_* \downarrow & & \downarrow \alpha_* \\ H_0^G(G/K; k) & \xrightarrow[\cong]{\gamma} & k(G/K) \end{array} \quad (3)$$

commutes.

The meaning of the other equivariant Eilenberg–Steenrod axioms is clear, they are natural equivariant versions of the first six ordinary Eilenberg–Steenrod axioms. It is the seventh axiom, the equivariant dimension axiom, described above, that is the key one here.

Note that each G -space of the form G/H , where $H < G$, should be considered as an “equivariant point” in the G -equivariant category, since the only non-empty G -invariant subset of G/H is G/H itself. Hence, for an equivariant ordinary homology theory, condition (1) should be satisfied. Given any equivariant homology theory $h_*^G(\cdot)$, the restriction of $h_*^G(\cdot)$ to the category of all G -spaces of the form G/H , where $H < G$, and G -homotopy classes of G -maps between such G -spaces, is called the *coefficients* of the theory $h_*^G(\cdot)$. Conditions (2) and (3) then express the fact that there exists a natural isomorphism between the coefficients of the equivariant homology theory $H_*^G(\cdot; k)$ and the given covariant coefficient system k . We call the theory $H_*^G(\cdot; k)$, constructed in [4], *equivariant singular homology with coefficients in k* .

Let X be a G -space. We shall now give a quick description of the construction of the n th, $n \geq 0$, equivariant singular homology group, in fact a left R -module, $H_n^G(X; k)$ of X , with coefficients in k . We refer to [4] for more details. An *equivariant singular n -simplex in X* is a G -equivariant map

$$T : \Delta_n \times G/H \rightarrow X,$$

where Δ_n , $n \geq 0$, denotes the standard n -simplex, and $H < G$. The topological group G acts on $\Delta_n \times G/H$ by acting trivially on Δ_n and on G/H in the standard way by multiplication from the left. We call H the type of T , and denote $t(T) = H$. Let for the moment $n \geq 0$ be fixed, and let $\mathcal{S} = \mathcal{S}_n(X)$ denote the set of all equivariant singular n -simplexes in X . Suppose k is a covariant coefficient system for G , over a ring R . We define

$$\hat{S}_n^G(X; k) = \bigoplus_{T \in \mathcal{S}} (\mathbb{Z}_T \otimes k(G/t(T))), \quad (4)$$

where the direct sum is over the set \mathcal{S} of all equivariant singular n -simplexes in X . Here \mathbb{Z}_T denotes the infinite cyclic group on the generator T , and each tensor product is over the integers. The left R -module structure on $k(G/t(T))$ makes $\mathbb{Z}_T \otimes k(G/t(T))$ into a left R -module, such that the map $i : k(G/t(T)) \rightarrow \mathbb{Z}_T \otimes k(G/t(T))$, $a \mapsto T \otimes a$, is an isomorphism of left R -modules. Thus each $\hat{S}_n^G(X; k)$, $n \geq 0$, is a left R -module.

Next we define a boundary homomorphism

$$\hat{\partial}_n : \hat{S}_n^G(X; k) \rightarrow \hat{S}_{n-1}^G(X; k) \quad (5)$$

by setting $\hat{\partial}_n(T \otimes a) = \sum_{i=0}^n (-1)^i (T^{(i)} \otimes a)$. Here $T^{(i)} = T \circ (e^i \times \text{id}) : \Delta_{n-1} \times G/t(T) \rightarrow X$ is the i th face of T , where $e^i : \Delta_{n-1} \rightarrow \Delta_n$ denotes the standard i th face map of Δ_n , $0 \leq i \leq n$. Then $\hat{\partial}_{n-1} \circ \hat{\partial}_n = 0$, and thus we obtain a chain complex

$$\hat{S}^G(X; k) = \{\hat{S}_n^G(X; k), \hat{\partial}_n\}. \quad (6)$$

Let $H, K < G$, and suppose $\eta : \Delta_n \times G/H \rightarrow \Delta_n \times G/K$ is a G -map, which covers id_{Δ_n} . Then there is a well-defined induced homomorphism

$$\eta_* : k(G/H) \rightarrow k(G/K),$$

see Lemma I.3.4 in [4]. Now let $T : \Delta_n \times G/H \rightarrow X$ and $T' : \Delta_n \times G/K \rightarrow X$ be two equivariant singular n -simplexes in X , and let $a \in k(G/H)$ and $b \in k(G/K)$. We set

$$T \otimes a \sim T' \otimes b \quad (7)$$

if there exists a G -map $\eta : \Delta_n \times G/H \rightarrow \Delta_n \times G/K$, which covers id_{Δ_n} , such that the diagram

$$\begin{array}{ccc} \Delta_n \times G/H & & \\ \eta \downarrow & \searrow T & \\ & & X \\ \Delta_n \times G/K & \nearrow T' & \end{array} \quad (8)$$

commutes, and $\eta_*(a) = b$.

By $\bar{S}_n^G(X; k)$ we denote submodule of $\hat{S}_n^G(X; k)$ generated by all elements of the form $T \otimes a - T' \otimes b$, where $T \otimes a \sim T' \otimes b$. We then define

$$S_n^G(X; k) = \hat{S}_n^G(X; k) / \bar{S}_n^G(X; k). \quad (9)$$

We let $p : \hat{S}_n^G(X; k) \rightarrow S_n^G(X; k)$ be the natural projection, and we also denote $p(c) = [c]$. The boundary homomorphism $\hat{\partial}_n$ induces a boundary homomorphism $\partial_n : S_n^G(X; k) \rightarrow S_{n-1}^G(X; k)$, defined by $\partial_n([c]) = [\hat{\partial}_n(c)]$, for every $c \in \hat{S}_n^G(X; k)$. Thus we obtain a chain complex

$$S^G(X; k) = \{S_n^G(X; k), \partial_n\}, \quad (10)$$

and we define

$$H_n^G(X; k) = \text{Ker } \partial_n / \text{Im } \partial_{n+1}, \quad \text{for each } n \geq 0. \quad (11)$$

That is, $H_n^G(X; k)$ is the n th homology group of the chain complex $S^G(X; k)$. We call $H_n^G(X; k)$ the n th equivariant singular homology group of the G -space X , with coefficients in k . Note that each homology group $H_n^G(X; k)$, $n \geq 0$, is in fact a left R -module.

Suppose (X, A) is a G -pair. Then the inclusion $i : A \rightarrow X$ induces a chain map

$$i_\# : S^G(A; k) \rightarrow S^G(X; k),$$

and it is easy to see that $i_\#$ is a monomorphism of chain complexes. We identify $S_n^G(A; k)$ with $i_\#(S_n^G(A; k))$ and consider $S_n^G(A; k)$ as a submodule of $S_n^G(X; k)$, for each $n \geq 0$. In this way $S^G(A; k)$ becomes a subcomplex of $S^G(X; k)$, and we define $S^G(X, A; k)$ to be the quotient chain complex

$$S^G(X, A; k) = S^G(X; k) / S^G(A; k). \quad (12)$$

The n th equivariant singular homology group $H_n^G(X, A; k)$, with coefficients in k , of the G -pair (X, A) , is then defined to be the n th homology group of the chain complex $S^G(X, A; k)$.

The short exact sequence

$$0 \rightarrow S_n^G(A; k) \xrightarrow{i_\#} S_n^G(X; k) \xrightarrow{j_\#} S_n^G(X, A; k) \rightarrow 0 \quad (13)$$

of chain complexes gives us a connecting homomorphism $\Delta : H_n^G(X, A; k) \rightarrow H_{n-1}^G(A; k)$, for each $n \geq 1$, and a long exact homology sequence

$$\cdots \xrightarrow{\Delta} H_n^G(A; k) \xrightarrow{i_*} H_n^G(X; k) \xrightarrow{j_*} H_n^G(X, A; k) \xrightarrow{\Delta} H_{n-1}^G(A; k) \xrightarrow{i_*} \cdots \quad (14)$$

We conclude this section with some remarks concerning the chain complex $S^G(X, A; k)$. We shall first consider the absolute case, so let X be a G -space. Let for the moment $n \geq 0$ be fixed, and let $\mathcal{S} = \mathcal{S}_n(X)$ denote the set of all equivariant singular n -simplexes in X . We define a relation \vdash in \mathcal{S} as follows. Suppose $T : \Delta_n \times G/H \rightarrow X$ and $T' : \Delta_n \times G/K \rightarrow X$ are equivariant singular n -simplexes in X . We define $T \vdash T'$ to hold if there exists a G -map $\eta : \Delta_n \times G/H \rightarrow \Delta_n \times G/K$, which covers id_{Δ_n} , such that the diagram

$$\begin{array}{ccc} \Delta_n \times G/H & & \\ \eta \downarrow & \searrow T & \\ \Delta_n \times G/K & \nearrow T' & X \end{array} \quad (15)$$

commutes. By \sim we denote the equivalence relation in $\mathcal{S} = \mathcal{S}_n(X)$ generated by the relation \vdash . Let $\mathcal{S}^* = \mathcal{S}_n^*(X) = \mathcal{S}_n(X)/\sim$ denote the set of equivalence classes under the equivalence relation \sim . If $T \in \mathcal{S}$ we let $[T]$ denote the equivalence class of T . We also use a script letter \mathcal{T} to denote such an equivalence class when we need to avoid the notation $[T]$, which singles out one specific representative T for the equivalence class $[T] = \mathcal{T}$.

Suppose k is a covariant coefficient system for G , over a ring R . Then our definition in (4) gives us

$$\hat{S}_n^G(X; k) = \bigoplus_{T \in \mathcal{S}} (\mathbb{Z}_T \otimes k(G/t(T))) = \bigoplus_{T \in \mathcal{S}^*} \bigoplus_{T \in \mathcal{T}} (\mathbb{Z}_T \otimes k(G/t(T))).$$

For each equivalence class $\mathcal{T} \in \mathcal{S}^*$ we denote

$$\hat{S}_n^G(X; k)(\mathcal{T}) = \bigoplus_{T \in \mathcal{T}} \mathbb{Z}_T \otimes k(G/t(T)). \quad (16)$$

Then

$$\hat{S}_n^G(X; k) = \bigoplus_{\mathcal{T} \in \mathcal{S}^*} \hat{S}_n^G(X; k)(\mathcal{T}).$$

Note that if $T \otimes a \sim T' \otimes b \in \hat{S}_n^G(X; k)$, then in particular $T \sim T'$, and hence $[T] = [T'] \in \mathcal{S}^*$. For each $\mathcal{T} \in \mathcal{S}^*$ we let $\tilde{S}_n^G(X; k)(\mathcal{T})$ denote the submodule of $\hat{S}_n^G(X; k)$ generated by all elements of the form $T \otimes a - T' \otimes b$, where $T \otimes a \sim T' \otimes b$, and where $T \in \mathcal{T}$ and hence also $T' \in \mathcal{T}$. Then

$$\tilde{S}_n^G(X; k)(\mathcal{T}) = \tilde{S}_n^G(X; k) \cap \hat{S}_n^G(X; k)(\mathcal{T}),$$

and hence $\tilde{S}_n^G(X; k) = \bigoplus_{\mathcal{T} \in \mathcal{S}^*} \tilde{S}_n^G(X; k)(\mathcal{T})$. We now define

$$S_n^G(X; k)(\mathcal{T}) = \hat{S}_n^G(X; k)(\mathcal{T}) / \tilde{S}_n^G(X; k)(\mathcal{T}) \quad (17)$$

and obtain

$$S_n^G(X; k) = \bigoplus_{\mathcal{T} \in \mathcal{S}^*} S_n^G(X; k)(\mathcal{T}). \quad (18)$$

Now, suppose that (X, A) is any G -pair. By $\mathcal{S}(X, A) = \mathcal{S}_n(X, A)$ we denote the set of all equivariant singular n -simplexes T in X for which $\text{Im}(T) \cap (X - A) \neq \emptyset$. Thus $\mathcal{S}(X, A) = \mathcal{S}(X) - \mathcal{S}(A)$. The equivalence relation \sim in $\mathcal{S}(X)$ induces an equivalence relation in $\mathcal{S}(A)$ and also in $\mathcal{S}(X, A)$, both of which we denote by \sim . We set $\mathcal{S}^*(X, A) = \mathcal{S}(X, A)/\sim$. Thus $\mathcal{S}^*(X, A) = \mathcal{S}^*(X) - \mathcal{S}^*(A)$. Then we have that

$$S_n^G(X, A; k) = \bigoplus_{\mathcal{T} \in \mathcal{S}^*(X, A)} S_n^G(X; k)(\mathcal{T}). \quad (19)$$

1.2. The maximal type and the canonical maximal representative of an equivariant singular n -simplex

Let X be a G -space, where G is a topological group, and let $T : \Delta_n \times G/t(T) \rightarrow X$ be an equivariant singular n -simplex in X , of type $t(T)$. We define

$$\tilde{t}(T) = \bigcap_{x \in D} G_x, \quad (1)$$

where $D = T(\Delta_n \times \{e(t(T))\})$. Thus $\tilde{t}(T)$ is a closed subgroup of G . Since T is a G -equivariant map we have that $t(T) \subset G_x$, for every $x \in D$, and hence

$$t(T) \subset \tilde{t}(T). \quad (2)$$

We call $\tilde{t}(T)$ the *maximal type* of T . Note that

$$\tilde{t}(T) \subset G_x, \quad \text{for each } x \in D. \quad (3)$$

Next we define a G -map

$$\bar{T} : \Delta_n \times G/\tilde{t}(T) \rightarrow X, \quad (v, g\tilde{t}(T)) \mapsto gT(v, et(T)). \quad (4)$$

The fact that \bar{T} is well defined is seen as follows. Let $v \in \Delta_n$ and denote $T(v, e(t(T))) = x$. Suppose $(v, g\tilde{t}(T)) = (v, g'\tilde{t}(T)) \in \Delta_n \times G/\tilde{t}(T)$. Then we have by (3) that $(g')^{-1}g \in \tilde{t}(T) \subset G_x$. Thus $(g')^{-1}gx = x$, which gives us $gT(v, et(T)) = g'T(v, et(T))$. This shows that \bar{T} is well defined, and clearly \bar{T} is G -equivariant. Furthermore \bar{T} is continuous, which is seen as follows. We have the commutative diagram

$$\begin{array}{ccc} \Delta_n \times G/t(T) & & \\ \downarrow \text{id} \times \beta & \searrow T & \\ \Delta_n \times G/\tilde{t}(T) & \nearrow \bar{T} & X \end{array} \quad (5)$$

Here $\beta : G/t(T) \rightarrow G/\tilde{t}(T)$, $g\tilde{t}(T) \mapsto g\tilde{t}(T)$, denotes the natural projection, cf. (2). The canonical projections $\pi : G \rightarrow G/t(T)$ and $\bar{\pi} : G \rightarrow G/\tilde{t}(T)$ are open surjective maps, and this implies that β is an open surjective map. Thus $\text{id} \times \beta$ is an open surjective map, and hence in particular a quotient map. Since T is continuous it now follows, by (5), that \bar{T} is continuous.

Thus $\bar{T} : \Delta_n \times G/\tilde{t}(T) \rightarrow X$ in (4) is an equivariant singular n -simplex in X , and

$$t(\bar{T}) = \tilde{t}(T), \quad (6)$$

i.e., the type of \bar{T} equals the maximal type of T . We call \bar{T} the *canonical maximal representative* of T . Note that $T \frown \bar{T}$, since $\bar{T} \circ (\text{id} \times \beta) = T$, and hence

$$[T] = [\bar{T}] \in \mathcal{S}^* = \mathcal{S}_n^*(X).$$

We say that an equivariant singular n -simplex T in X is *maximal* if $T = \bar{T}$.

1.3. Some further notation and definitions

Let G be a topological group and let X be a G -space. Suppose

$$T : \Delta_n \times G/H \rightarrow X$$

is an equivariant singular n -simplex, of type H , in X . For each $g \in G$ we define

$$T_g : \Delta_n \rightarrow X, \quad v \mapsto T(v, gH). \quad (1)$$

Then $T_g = T_{gh}$, for every $h \in H = t(T)$. By

$$T^* : \Delta_n \rightarrow X/G, \quad v \mapsto \pi(T(v, eH)), \quad (2)$$

we denote the map induced by T in the orbit spaces. Here $\pi : X \rightarrow X/G$ denotes the natural projection onto the orbit space X/G . Note that $T^* = \pi \circ T_g$, for every $g \in G$.

Now suppose $F : \Delta_n \rightarrow X$ is an ordinary singular n -simplex in X . We define

$$\langle F \rangle : \Delta_n \times G \rightarrow X, \quad (v, g) \mapsto gF(v). \quad (3)$$

Then $\langle F \rangle$ is an equivariant singular n -simplex, of type $\{e\}$, in X . Clearly

$$\langle F \rangle_e = F. \quad (4)$$

Suppose that

$$D : \Delta_n \rightarrow X/G$$

is a singular n -simplex in the orbit space X/G , and assume that there exists a lifting

$$F : \Delta_n \rightarrow X$$

of the map D to X , i.e., $D = \pi \circ F$. Then $\langle F \rangle^*(v) = \pi(\langle F \rangle(v, e)) = \pi(F(v)) = D(v)$, for all $v \in \Delta_n$. Thus

$$\langle F \rangle^* = D. \quad (5)$$

Let

$$T : \Delta_n \times G \rightarrow X \quad (6)$$

be an equivariant singular n -simplex, of type $\{e\}$, in X . Then $\langle T_e \rangle(v, g) = gT_e(v) = gT(v, e) = T(v, g)$, for all $(v, g) \in \Delta_n \times G$. Thus we have in this case that

$$\langle T_e \rangle = T. \quad (7)$$

2. The chain complex $S^G(X, A; k)$ in the case of a totally disconnected locally compact group G

The main purpose of this section is to prove Theorem 2.6, and its Corollary 2.7. In order to do this we first need some preliminary facts, which we give in the lemmas and corollaries, 2.1 to 2.5, below. Throughout this section G denotes a totally disconnected, locally compact group.

We say that two equivariant singular n -simplexes T and T' , in a G -space in X , are *isomorphic* if there exists a G -homeomorphism $\eta : \Delta_n \times G/t(T) \rightarrow \Delta_n \times G/t(T')$, which covers id_{Δ_n} , such that $T = T' \circ \eta$.

Lemma 2.1. *Let G be a totally disconnected locally compact group, and let $H, K < G$. Suppose $\eta : \Delta_n \times G/H \rightarrow \Delta_n \times G/K$ is a G -map, which covers id_{Δ_n} . Then there exists $g_0K \in G/K$ such that $\eta(v, gH) = (v, gg_0K)$, for every $(v, gH) \in \Delta_n \times G/H$.*

Proof. The space G/K is totally disconnected, by Proposition A.1, and $p_2(\eta(\Delta_n \times \{eH\}))$ is a connected subset of G/K , since Δ_n is connected. Here $p_2 : \Delta_n \times G/K \rightarrow G/K$ denotes the projection onto the second factor. Thus $p_2(\eta(\Delta_n \times \{eH\})) = \{g_0K\}$, for some $g_0K \in G/K$, and hence $\eta(v, eH) = (v, g_0K)$, for every $v \in \Delta_n$. Since η is a G -map, we have that $\eta(v, gH) = g\eta(v, eH) = g(v, g_0K) = (v, gg_0K)$, for every $(v, gH) \in \Delta_n \times G/H$. \square

Lemma 2.2. *Let G be a totally disconnected locally compact group, and let X be any G -space. Suppose T and T' are equivariant singular n -simplexes in X , such that $T \sim T'$. Then the canonical maximal representatives \bar{T} and \bar{T}' are isomorphic.*

Proof. Since the equivalence relation \sim in $\mathcal{S} = \mathcal{S}_n(X)$ is generated by the relation \vdash , we may assume that $T \vdash T'$. Then there exists a G -map $\eta : \Delta_n \times G/t(T) \rightarrow \Delta_n \times G/t(T')$, which covers id_{Δ_n} , such that the diagram

$$\begin{array}{ccc} \Delta_n \times G/t(T) & & \\ \eta \downarrow & \searrow T & \\ & & X \\ & \nearrow T' & \\ \Delta_n \times G/t(T') & & \end{array}$$

commutes. Let us denote $t(T) = H$ and $t(T') = K$.

By Lemma 2.1 there exists $g_0K \in G/K$, such that $\eta(v, eH) = (v, g_0K)$, for all $v \in \Delta_n$. Then $T_e(v) = T(v, eH) = T'(v, g_0K) = g_0T'(v, eK) = g_0T'_e(v)$, for all $v \in \Delta_n$, i.e.,

$$T_e = g_0T'_e : \Delta_n \rightarrow X.$$

We denote $D = T_e(\Delta_n)$ and $D' = T'_e(\Delta_n)$. Then $D = g_0 D'$, and hence it follows that

$$\tilde{t}(T) = \bigcap_{x \in D} G_x = \bigcap_{y \in D'} G_{g_0 y} = \bigcap_{y \in D'} (g_0 G_y g_0^{-1}) = g_0 \left(\bigcap_{y \in D'} G_y \right) g_0^{-1} = g_0 \tilde{t}(T') g_0^{-1}. \quad (1)$$

Thus we obtain a commutative diagram

$$\begin{array}{ccc} \Delta_n \times G/\tilde{t}(T) & & \\ \downarrow \bar{\eta} & \searrow \bar{T} & \\ & & X \\ & \nearrow \bar{T}' & \\ \Delta_n \times G/\tilde{t}(T') & & \end{array}$$

where $\bar{\eta}$ is given by $(v, g\tilde{t}(T)) \mapsto (v, gg_0\tilde{t}(T'))$, and \bar{T} and \bar{T}' are the canonical maximal representatives of T and T' , respectively, see (4) in Section 1.2. Clearly $\bar{\eta}$ is a G -map, which covers id_{Δ_n} , and it follows by (1) that $\bar{\eta}$ is a G -homeomorphism. Thus \bar{T} and \bar{T}' are isomorphic. \square

Corollary 2.3. Let G and X be as in Lemma 2.2. Suppose U and U' are two maximal equivariant singular n -simplexes in X such that $U \frown U'$. Then U and U' are isomorphic.

Proof. Since $U = \bar{U}$ and $U' = \bar{U}'$, the corollary follows directly by Lemma 2.2. \square

Lemma 2.4. Let G and X be as in Lemma 2.2. Suppose T and U are equivariant singular n -simplexes in X , and assume that U is maximal. Furthermore suppose that there are G -maps $\eta_1, \eta_2 : \Delta_n \times G/t(T) \rightarrow \Delta_n \times G/t(U)$, over id_{Δ_n} , such that the diagrams

$$\begin{array}{ccc} \Delta_n \times G/t(T) & & \\ \downarrow \eta_i & \searrow T & \\ & & X \\ & \nearrow U & \\ \Delta_n \times G/t(U) & & \end{array}$$

$i = 1$ and 2 commute. Then $\eta_1 = \eta_2$.

Proof. We denote $t(T) = H$ and $t(U) = K$. By Lemma 2.1 there exist $g_i K \in G/K$, $i = 1$ and 2 , such that $\eta_i(v, gH) = (v, gg_i K)$, for every $(v, gH) \in \Delta_n \times G/H$. Then $T_e(v) = T(v, eH) = (U \circ \eta_i)(v, eH) = U(v, g_i K) = g_i U(v, eK) = g_i U_e(v)$, for $i = 1$ and 2 , and every $v \in \Delta_n$. Thus $g_1 U_e = g_2 U_e : \Delta_n \rightarrow X$, and denoting $g_0 = g_1^{-1} g_2$ we have that

$$U_e = g_0 U_e : \Delta_n \rightarrow X.$$

Let $v \in \Delta_n$ and denote $U_e(v) = x$. Then $x = U_e(v) = (g_0 U_e)(v) = g_0 x$. Thus $g_0 \in G_x$, for every $x \in U_e(\Delta_n) = D$, and therefore

$$g_0 \in \bigcap_{x \in D} G_x = \tilde{t}(U) = t(U) = K.$$

Thus $g_0 K = eK$, and hence $g_1 K = g_2 K \in G/K$. It now follows that $\eta_1 = \eta_2$. \square

Corollary 2.5. Let G and X be as in Lemma 2.2, and let T and U be equivariant singular n -simplexes in X , such that $T \frown U$, and assume that U is maximal. Then there exists a unique G -map $\theta : \Delta_n \times G/t(T) \rightarrow \Delta_n \times G/t(U)$, over id_{Δ_n} , such that the diagram

$$\begin{array}{ccc} \Delta_n \times G/t(T) & & \\ \downarrow \theta & \searrow T & \\ & & X \\ & \nearrow U & \\ \Delta_n \times G/t(U) & & \end{array}$$

commutes.

Proof. Let \bar{T} be the canonical maximal representative of T . Then $T = \bar{T} \circ (\text{id} \times \beta)$, where $\beta : G/t(T) \rightarrow G/\bar{t}(T)$ denotes the natural projection, see (5) in Section 1.2. Now $\bar{T} \frown T \frown U$, and since both \bar{T} and U are maximal there exists, by Corollary 2.3, a G -homeomorphism $\zeta : \Delta_n \times G/t(\bar{T}) \rightarrow \Delta_n \times G/t(U)$, over id_{Δ_n} , such that $\bar{T} = U \circ \zeta$. Then $T = U \circ \zeta \circ (\text{id} \times \beta)$, and thus $\theta = \zeta \circ (\text{id} \times \beta) : \Delta_n \times G/t(T) \rightarrow \Delta_n \times G/t(U)$ is a G -map, over id_{Δ_n} , such that $T = U \circ \theta$. The uniqueness follows by Lemma 2.4. \square

Theorem 2.6. Let X be a G -space, where G is a totally disconnected locally compact group, and let k be any covariant coefficient system for G , over a ring R . Suppose $T \in S^* = S_n^*(X)$, $n \geq 0$, and let $U \in \mathcal{T}$ be a maximal equivariant singular n -simplex in X representing the equivalence class T . Then

$$S_n^G(X; k)(T) \cong \mathbb{Z}_U \otimes k(G/t(U)) \cong k(G/t(U)).$$

Proof. We shall show that there exists an isomorphism

$$q : S_n^G(X; k)(T) \xrightarrow{\cong} \mathbb{Z}_U \otimes k(G/t(U)). \quad (1)$$

First we define a homomorphism, of left R -modules,

$$\hat{q} : S_n^G(X; k)(T) \rightarrow \mathbb{Z}_U \otimes k(G/t(U)), \quad \text{for each } n \geq 0. \quad (2)$$

Let $T \otimes a \in \hat{S}_n^G(X; k)(T)$. Since $T \frown U$ and U is maximal there exists by Corollary 2.5 a unique G -map $\theta : \Delta_n \times G/t(T) \rightarrow \Delta_n \times G/t(U)$, over id_{Δ_n} , such that

$$T = U \circ \theta. \quad (3)$$

We define

$$\hat{q}(T \otimes a) = U \otimes \theta_*(a), \quad (4)$$

where $\theta_* : k(G/t(T)) \rightarrow k(G/t(U))$. By extending linearly this gives us the homomorphism \hat{q} in (2). Note that \hat{q} is surjective, since if $U \otimes b \in \mathbb{Z}_U \otimes k(G/t(U))$, then $\hat{q}(U \otimes b) = U \otimes b$.

Now suppose that $T \otimes a \sim T' \otimes b \in \hat{S}_n^G(X; k)(T)$. Then there exists a G -map $\eta : \Delta_n \times G/t(T) \rightarrow \Delta_n \times G/t(T')$, over id_{Δ_n} , such that $T = T' \circ \eta$ and $\eta_*(a) = b$. We have that $T' \frown U$, and we let θ be as in (3) and $\theta' : \Delta_n \times G/t(T') \rightarrow \Delta_n \times G/t(U)$ be the unique G -map, over id_{Δ_n} , given by Corollary 2.5, for which $T' = U \circ \theta'$. Then $T = T' \circ \eta = U \circ \theta' \circ \eta$, and hence we have by the uniqueness part of Corollary 2.5 that

$$\theta = \theta' \circ \eta.$$

Therefore $\hat{q}(T \otimes a) = U \otimes \theta_*(a) = U \otimes \theta'_*(\eta_*(a)) = U \otimes \theta'_*(b) = \hat{q}(T' \otimes b)$, and hence $\hat{q}(T \otimes a - T' \otimes b) = 0$. Thus the homomorphism \hat{q} in (2) induces a homomorphism,

$$q : S_n^G(X; k)(T) \rightarrow \mathbb{Z}_U \otimes k(G/t(U)), \quad [c] \mapsto \hat{q}(c). \quad (5)$$

Furthermore q is surjective, since \hat{q} is surjective.

It remains to show that q is injective. Let $[c], [c'] \in S_n^G(X; k)(T)$ be such that $q([c]) = q([c'])$. Here $c = \sum_{i=1}^r T_i \otimes a_i$, where $T_i \in \mathcal{T}$ and $a_i \in k(G/t(T_i))$, $1 \leq i \leq r$, and $c' = \sum_{j=1}^s T'_j \otimes a'_j$, where $T'_j \in \mathcal{T}$ and $a'_j \in k(G/t(T'_j))$, $1 \leq j \leq s$. By Corollary 2.4 there exist uniquely determined G -maps, over id_{Δ_n} , $\theta_i : \Delta_n \times G/t(T_i) \rightarrow \Delta_n \times G/t(U)$, such that $T_i = U \circ \theta_i$, $1 \leq i \leq r$, and $\theta'_j : \Delta_n \times G/t(T'_j) \rightarrow \Delta_n \times G/t(U)$, such that $T'_j = U \circ \theta'_j$, $1 \leq j \leq s$. Now

$$q([c]) = \hat{q}(c) = \hat{q}\left(\sum_{i=1}^r T_i \otimes a_i\right) = \sum_{i=1}^r \hat{q}(T_i \otimes a_i) = \sum_{i=1}^r U \otimes (\theta_i)_*(a_i) \quad (6)$$

and

$$q([c']) = \hat{q}(c') = \hat{q}\left(\sum_{j=1}^s T'_j \otimes a'_j\right) = \sum_{j=1}^s \hat{q}(T'_j \otimes a'_j) = \sum_{j=1}^s U \otimes (\theta'_j)_*(a'_j). \quad (7)$$

Since $T_i \otimes a_i \sim U \otimes (\theta_i)_*(a_i)$, for $1 \leq i \leq r$, we have that

$$\sum_{i=1}^r (T_i \otimes a_i - U \otimes (\theta_i)_*(a_i)) \in \tilde{S}_n^G(X; k)(T), \quad (8)$$

and since $T'_j \otimes a'_j \sim U \otimes (\theta'_j)_*(a'_j)$, we have that

$$\sum_{j=1}^s (T'_j \otimes a'_j - U \otimes (\theta'_j)_*(a'_j)) \in \tilde{S}_n^G(X; k)(\mathcal{T}). \quad (9)$$

Now the assumption $q([c]) = q([c'])$, together with (6)–(9), imply that

$$c - c' = \sum_{i=1}^r T_i \otimes a_i - \sum_{j=1}^s T'_j \otimes a'_j \in \tilde{S}_n^G(X; k)(\mathcal{T}).$$

Thus $[c] = [c'] \in S_n^G(X; k)(\mathcal{T})$, and this shows that q is injective, and hence q is an isomorphism. \square

Corollary 2.7. Suppose G is a totally disconnected locally compact group, and let (X, A) be any G -pair. Let k be a covariant coefficient system for G , over a ring R . Then, for each $n \geq 0$, we have that

$$S_n^G(X, A; k) \cong \bigoplus_{[U] \in S^*(X, A)} \mathbb{Z}_{\bar{U}} \otimes k(G/t(\bar{U})).$$

Here the direct sum is over the set $S^*(X, A)$ of equivalence classes of equivariant singular n -simplexes in (X, A) .

Proof. Follows by Theorem 2.6, and (19) in Section 1.1. \square

3. Change of coefficient systems and a long exact homology sequence

To begin with we let G denote any topological group. Suppose that k_1 and k_2 are covariant coefficient systems for G , over a ring R , and let

$$\mu : k_1 \rightarrow k_2 \quad (1)$$

be a natural transformation of covariant coefficient systems. This means that, for each $H < G$, we have a homomorphism

$$\mu : k_1(G/H) \rightarrow k_2(G/H) \quad (2)$$

of left R -modules, and if $\alpha : G/H \rightarrow G/K$, where $H, K < G$, is a G -map (representing a G -homotopy class of G -maps from G/H to G/K) then the diagram

$$\begin{array}{ccc} k_1(G/H) & \xrightarrow{\mu} & k_2(G/H) \\ \alpha_* \downarrow & & \downarrow \alpha_* \\ k_1(G/K) & \xrightarrow{\mu} & k_2(G/K) \end{array} \quad (3)$$

commutes.

Let X be a G -space, and let $n \geq 0$. Given a natural transformation $\mu : k_1 \rightarrow k_2$ of coefficient systems, we obtain a homomorphism, of left R -modules,

$$\hat{\mu}_\# : \hat{S}_n^G(X; k_1) \rightarrow \hat{S}_n^G(X; k_2) \quad (4)$$

if we for each element of the form $T \otimes a$, define

$$\hat{\mu}_\#(T \otimes a) = T \otimes \mu(a) \quad (5)$$

and then extend linearly. Here $T \in S = S_n(X)$, $a \in k_1(G/t(T))$ and $\mu : k_1(G/t(T)) \rightarrow k_2(G/t(T))$. It is clear from the above definition that $\partial \circ \hat{\mu}_\# = \hat{\mu}_\# \circ \partial$, and thus the homomorphisms $\hat{\mu}_\#$ in (4), for all $n \geq 0$, form a chain map $\hat{\mu}_\# : \hat{S}^G(X; k_1) \rightarrow \hat{S}^G(X; k_2)$.

If $T \otimes a \sim T' \otimes b \in \hat{S}_n^G(X; k_1)$ there exists a G -map $\eta : \Delta_n \times G/t(T) \rightarrow \Delta_n \times G/t(T')$, which covers id_{Δ_n} , such that $T = T' \circ \eta$ and $\eta_*(a) = b$. By (3) we have $\eta_*(\mu(a)) = \mu(\eta_*(a)) = \mu(b)$. Therefore $\hat{\mu}_\#$ in (4) induces a homomorphism

$$\mu_\# : S_n^G(X; k_1) \rightarrow S_n^G(X; k_2), \quad (6)$$

for each $n \geq 0$. Thus we obtain a chain map $\mu_\# : S^G(X; k_1) \rightarrow S^G(X; k_2)$, which in turn induces homomorphisms, of left R -modules,

$$\mu_* : H_n^G(X; k_1) \rightarrow H_n^G(X; k_2), \quad \text{for all } n \geq 0. \quad (7)$$

Now suppose that k_0, k_1 and k_2 are covariant coefficient systems for G , over a ring R , and let

$$0 \rightarrow k_0 \xrightarrow{\lambda} k_1 \xrightarrow{\mu} k_2 \rightarrow 0 \quad (8)$$

be a short exact sequence of coefficient systems, i.e., λ and μ are natural transformations of coefficient systems and the sequence

$$0 \rightarrow k_0(G/H) \xrightarrow{\lambda} k_1(G/H) \xrightarrow{\mu} k_2(G/H) \rightarrow 0 \quad (9)$$

is a short exact sequence, of left R -modules, for each $H < G$.

We should here point out that due to the definition $S_n^G(X; k) = \hat{S}_n^G(X; k)/\bar{S}_n^G(X; k)$ of the chain groups $S_n^G(X; k)$, $n \geq 0$, see (9) in Section 1.1, it is when G is an arbitrary topological group *not at all* clear that a short exact sequence of coefficient systems, as in (8), should induce a short exact sequence of the chain groups $S_n^G(X; k_i)$, $i = 0, 1$ and 2 . However, when G is a totally disconnected locally compact group we obtain as a consequence of Theorem 2.6 and its Corollary 2.7 the following result.

Theorem 3.1. *Let G be a totally disconnected locally compact group, and let (X, A) be any G -pair. Suppose $0 \rightarrow k_0 \xrightarrow{\lambda} k_1 \xrightarrow{\mu} k_2 \rightarrow 0$ is a short exact sequence of covariant coefficient systems for G , over a ring R . Then the induced sequence of chain complexes*

$$0 \rightarrow S^G(X, A; k_0) \xrightarrow{\lambda_{\#}} S^G(X, A; k_1) \xrightarrow{\mu_{\#}} S^G(X, A; k_2) \rightarrow 0 \quad (*)$$

is exact. Hence we obtain a long exact sequence of equivariant singular homology groups

$$\cdots \xrightarrow{\Delta} H_n^G(X, A; k_0) \xrightarrow{\lambda_*} H_n^G(X, A; k_1) \xrightarrow{\mu_*} H_n^G(X, A; k_2) \xrightarrow{\Delta} H_{n-1}^G(X, A; k_0) \xrightarrow{\lambda_*} \cdots$$

Proof. By Corollary 2.7 the sequence $(*)$ of chain complexes is in each degree n , $n \geq 0$, a direct sum of short exact sequences of the form

$$0 \rightarrow k_0(G/t(U)) \xrightarrow{\lambda} k_1(G/t(U)) \xrightarrow{\mu} k_2(G/t(U)) \rightarrow 0.$$

Hence $(*)$ is a short exact sequence of chain complexes. The second claim in Theorem 3.1 now follows by a standard result in homological algebra. \square

4. Equivariant singular homology with coefficients in the natural coefficient system Λ

In this section G denotes a totally disconnected locally compact group, and R is a ring. Our aim is to prove Theorem 4.1 below. First we define a covariant coefficient system Λ for G , over the ring R , as follows. For each $H < G$ we set

$$\Lambda(G/H) = R[G/H], \quad (1)$$

where $R[G/H]$ denotes the free left R -module on the set G/H . Suppose $\alpha : G/H \rightarrow G/K$ is a G -map, where $H, K < G$, and let us denote $\alpha(eH) = g_0K$. We then define

$$\alpha_* : \Lambda(G/H) \rightarrow \Lambda(G/K) \quad (2)$$

to be the map

$$\alpha_* : R[G/H] \rightarrow R[G/K], \quad \sum_{i=1}^m r_i(g_iH) \mapsto \sum_{i=1}^m r_i(g_i g_0K), \quad (3)$$

where $r_i \in R$, $g_iH \in G/H$ and $g_i g_0K \in G/K$, $1 \leq i \leq m$. Clearly α_* is a homomorphism of left R -modules. Suppose $\alpha, \beta : G/H \rightarrow G/K$ are two G -maps that are G -homotopic. Then $\alpha(eH) = \beta(eH)$, since G/K is totally disconnected, see Proposition A.1. Now $\alpha(gH) = g\alpha(eH) = g\beta(eH) = \beta(gH)$, for every $gH \in G/H$, and hence $\alpha = \beta$, and therefore also $\alpha_* = \beta_* : \Lambda(G/H) \rightarrow \Lambda(G/K)$. Thus Λ is a coefficient system, and we call Λ the *natural covariant coefficient system for G , over the ring R* .

Also note that since G/H is totally disconnected for each $H < G$, by Proposition A.1, we have that $\Lambda(G/H) \cong H_0(G/H; R)$, for each $H < G$, and furthermore α_* in (2) equals the induced homomorphism $\alpha_* : H_0(G/H; R) \rightarrow H_0(G/K; R)$. Here $H_0(\cdot; R)$ denotes ordinary singular homology, with coefficients in R , of degree 0.

Theorem 4.1. *Let G be a totally disconnected locally compact group, and let Λ be the natural covariant coefficient system for G over a ring R . Then there exists, for any G -space X , a natural isomorphism*

$$\Phi : H_n^G(X; \Lambda) \xrightarrow{\cong} H_n(X; R), \quad \text{for each } n \geq 0.$$

The same holds for any G -pair (X, A) in place of X . Here $H_*(\cdot; R)$ denotes ordinary singular homology, with coefficients in R .

Proof. In fact there exists an isomorphism already on the level of chain complexes, so we shall construct a chain map

$$\varphi : S^G(X; \Lambda) \rightarrow S(X; R), \quad (1)$$

and show that it is an isomorphism of chain complexes. Here $S(X; R)$ denotes the ordinary singular chain complex of the space X , with coefficients in R . For each $n \geq 0$ we define a homomorphism of left R -modules

$$\hat{\varphi} : \hat{S}_n^G(X; \Lambda) \rightarrow S_n(X; R) \quad (2)$$

as follows. The left R -module $\hat{S}_n^G(X; \Lambda)$ is a direct sum of left R -modules of the form $\mathbb{Z}_T \otimes \Lambda(G/H)$, where $H = t(T)$. Furthermore $\Lambda(G/H) = R[G/H]$ equals the free left R -module on the set G/H . Hence it is enough to define the value $\hat{\varphi}(T \otimes gH)$, for each equivariant singular n -simplex $T : \Delta_n \times G/H \rightarrow X$ and each $gH \in G/H$. We define

$$\hat{\varphi}(T \otimes gH) = T_g \otimes 1, \quad (3)$$

where $T_g : \Delta_n \rightarrow X$, $v \mapsto T(v, gH)$, is as defined in (1) in Section 1.3, and $1 \in R$. This gives us the homomorphism $\hat{\varphi}$ in (2). Note that if $a = \sum_{i=1}^m r_i(g_iH) \in \Lambda(G/H)$, then

$$\hat{\varphi}(T \otimes a) = \sum_{i=1}^m T_{g_i} \otimes r_i. \quad (4)$$

Clearly $(T^{(j)})_g = (T_g)^{(j)}$, for $0 \leq j \leq n$, and every $g \in G$. Here $T^{(j)}$ denotes the j th face of the equivariant singular n -simplex $T : \Delta_n \times G/H \rightarrow X$, and $(T_g)^{(j)}$ is the j th face of the ordinary singular n -simplex $T_g : \Delta_n \rightarrow X$. Therefore the homomorphisms $\hat{\varphi}$ in (2), for all n , satisfy $\partial \circ \hat{\varphi} = \hat{\varphi} \circ \partial$. Thus we obtain a chain map $\hat{\varphi} : \hat{S}^G(X; \Lambda) \rightarrow S(X; R)$.

Next we show that the homomorphism $\hat{\varphi}$ in (2) induces a homomorphism

$$\varphi : S_n^G(X; \Lambda) \rightarrow S_n(X; R). \quad (5)$$

Suppose $T \otimes a \sim T' \otimes b \in \hat{S}_n^G(X; \Lambda)$. We denote $t(T) = H$ and $t(T') = K$. Then there exists a G -map $\eta : \Delta_n \times G/H \rightarrow \Delta_n \times G/K$, which covers id_{Δ_n} , such that $T = T' \circ \eta$ and $\eta_*(a) = b$, where $\eta_* : \Lambda(G/H) \rightarrow \Lambda(G/K)$. By Lemma 2.1 there exists $g_0K \in G/K$, such that $\eta(v, gH) = (v, gg_0K)$, for every $(v, gH) \in \Delta_n \times G/H$. Therefore $T_g(v) = T(v, gH) = T'(\eta(v, gH)) = T'(v, gg_0K) = T'_{gg_0}(v)$, for each $g \in G$ and every $v \in \Delta_n$. Thus

$$T_g = T'_{gg_0} : \Delta_n \rightarrow X, \quad \text{for each } g \in G. \quad (6)$$

If $a = \sum_{i=1}^m r_i(g_iH)$, then $\eta_*(a) = b = \sum_{i=1}^m r_i(g_i g_0K)$. Hence we have by (4) and (6) that

$$\hat{\varphi}(T \otimes a) = \sum_{i=1}^m T_{g_i} \otimes r_i = \sum_{i=1}^m T'_{g_i g_0} \otimes r_i = \hat{\varphi}(T' \otimes b). \quad (7)$$

It follows that, for each $n \geq 0$, the homomorphism $\hat{\varphi}$ in (2) induces φ in (5), and this gives us a chain map $\varphi : S^G(X; \Lambda) \rightarrow S(X; R)$ as in (1).

Next we define a homomorphism of left R -modules

$$\psi : S_n(X; R) \rightarrow S_n^G(X; \Lambda) \quad (8)$$

for every $n \geq 0$, as follows. For each ordinary singular n -simplex $F : \Delta_n \rightarrow X$, we set

$$\psi(F \otimes 1) = p(\langle F \rangle \otimes e) = [\langle F \rangle \otimes e], \quad (9)$$

and then extend linearly. Here $\langle F \rangle : \Delta_n \times G \rightarrow X$, $(v, g) \mapsto gF(v)$, see (3) in Section 1.3, $1 \in R$ and $e = 1e \in R[G] = \Lambda(G)$. Moreover $p : S_n^G(X; \Lambda) \rightarrow S_n^G(X; \Lambda)$ denotes the projection, and we also denote $p(c) = [c]$.

We claim that $\varphi \circ \psi = \text{id}$. If $F : \Delta_n \rightarrow X$ is a singular n -simplex in X , then

$$(\varphi \circ \psi)(F \otimes 1) = \varphi([\langle F \rangle \otimes e]) = \hat{\varphi}(\langle F \rangle \otimes e) = \langle F \rangle_e \otimes 1 = F \otimes 1.$$

Here $\langle F \rangle_e = F$ is given by (4) in Section 1.3. Hence $\varphi \circ \psi = \text{id}$.

Next we show that $\psi \circ \varphi = \text{id}$. Let $T : \Delta_n \times G/H \rightarrow X$ be an equivariant singular n -simplex, of type H , in X , and let $gH \in G/H$. Then

$$(\psi \circ \varphi)([T \otimes gH]) = \psi(\hat{\varphi}(T \otimes gH)) = \psi(T_g \otimes 1) = [\langle T_g \rangle \otimes e].$$

We claim that

$$[\langle T_g \rangle \otimes e] = [T \otimes gH]. \quad (10)$$

Here $T_g : \Delta_n \rightarrow X$, $v \mapsto T(v, gH)$, see (1) in Section 1.3. Then $\langle T_g \rangle : \Delta_n \times G \rightarrow X$ is given by $\langle T_g \rangle(v, g') = g'T_g(v) = g'T(v, gH) = T(v, g'gH)$, for all $(v, g') \in \Delta_n \times G$, see (3) in Section 1.3. We consider the diagram

$$\begin{array}{ccc} \Delta_n \times G & & \\ \eta_g \downarrow & \searrow \langle T_g \rangle & \\ \Delta_n \times G/H & \xrightarrow{T} & X \end{array}$$

where $\eta_g(v, g') = (v, g'gH)$, for all $(v, g') \in \Delta_n \times G$. Then η_g is a G -map which covers id_{Δ_n} , and $T(\eta_g(v, g')) = T(v, g'gH)$. Thus the diagram commutes, and hence

$$\langle T_g \rangle \otimes e \sim T \otimes (\eta_g)_*(e) = T \otimes gH.$$

Therefore our claim in (11) holds, and hence $\psi \circ \varphi = \text{id}$. Thus φ in (5) is an isomorphism, for each $n \geq 0$, and hence the chain map φ in (1) is an isomorphism of chain complexes. \square

5. Free actions

The purpose of this section is to establish the result in Theorem 5.5. First we introduce some notation. Suppose G is a topological group acting on a Hausdorff space X , and let U and V be subsets of X . Then we denote

$$G[V, U] = \{g \in G \mid V \cap gU \neq \emptyset\}$$

and

$$G[U] = G[U, U].$$

Definition 5.1. Let X be a G -space, where G is a topological group. We say that the action of G on X is Cartan, or that X is a Cartan G -space, if each point $x \in X$ has a neighborhood U , such that $\overline{G[U]}$ is compact.

Note that if G is compact then every action of G is Cartan.

Lemma 5.2. Let G be a topological group and let X be a free Cartan G -space. Let $D : \Delta_n \rightarrow X/G$ be a singular n -simplex in the orbit space X/G , and suppose that $F : \Delta_n \rightarrow X$ and $F' : \Delta_n \rightarrow X$ are two liftings of D to X . Then there exists a continuous function $\kappa : \Delta_n \rightarrow G$ such that $F(v) = \kappa(v)F'(v)$, for all $v \in \Delta_n$.

Proof. For each $v \in \Delta_n$ the points $F(v)$ and $F'(v)$ belong to the same G -orbit in X . Since the action of G on X is free there exists a unique element $\kappa(v) \in G$ such that

$$F(v) = \kappa(v)F'(v). \quad (1)$$

This gives us a function

$$\kappa : \Delta_n \rightarrow G, \quad v \mapsto \kappa(v). \quad (2)$$

The fact that κ is continuous is seen as follows. Suppose $\bar{v} \in \Delta_n$ and let $v_i \in \Delta_n$, $i \in \mathbb{N}$, be a sequence of points in Δ_n , such that $\lim_{i \in \mathbb{N}} v_i = \bar{v}$. We claim that $\lim_{i \in \mathbb{N}} \kappa(v_i) = \kappa(\bar{v})$. Let us denote

$$\kappa(\bar{v}) = \hat{g}, \quad \text{and} \quad \kappa(v_i) = g_i,$$

for each $i \in \mathbb{N}$. Thus our claim is that

$$\lim_{i \in \mathbb{N}} g_i = \hat{g}. \quad (3)$$

In the following we denote

$$F(\bar{v}) = \bar{x}, \quad \text{and} \quad F'(\bar{v}) = \bar{x}',$$

and also

$$F(v_i) = x_i, \quad \text{and} \quad F'(v_i) = x'_i,$$

for each $i \in \mathbb{N}$. Then we have by (1) that

$$\bar{x} = \hat{g}\bar{x}'. \quad (4)$$

and also that

$$x_i = g_i x'_i, \quad (5)$$

for every $i \in \mathbb{N}$. The fact that F and F' are continuous gives us

$$\lim_{i \in \mathbb{N}} x_i = \bar{x}, \quad \text{and} \quad \lim_{i \in \mathbb{N}} x'_i = \bar{x}'. \quad (6)$$

We now choose an open neighborhood U of $\bar{x}' = F'(\bar{v})$ in X , such that $\overline{G[U]}$ is compact. Then $\hat{g}U$ is an open neighborhood of $\hat{g}\bar{x}' = \bar{x} = F(\bar{v})$ in X . Since F and F' are continuous there exists an open neighborhood V of \bar{v} in Δ_n , for which

$$F(V) \subset \hat{g}U, \quad \text{and} \quad F'(V) \subset U. \quad (7)$$

Since $\lim_{i \in \mathbb{N}} v_i = \bar{v}$, we may assume that $v_i \in V$, for all $i \in \mathbb{N}$. Then

$$x_i = g_i x'_i \in F(V) \cap g_i F'(V),$$

for every $i \in \mathbb{N}$. Hence $F(V) \cap g_i F'(V) \neq \emptyset$, for all $i \in \mathbb{N}$, and thus, using (7), we have that

$$g_i \in G[F(V), F'(V)] \subset G[\hat{g}U, U] = \hat{g}G[U, U] \subset \hat{g}\overline{G[U]}, \quad (8)$$

for all $i \in \mathbb{N}$. Furthermore $\hat{g}\overline{G[U]}$ is compact since $\overline{G[U]}$ is compact.

Now suppose that our claim in (3) does not hold, i.e., suppose that the sequence $\{g_i: i \in \mathbb{N}\}$ does not converge to \check{g} . Then there exists an open neighborhood W of \check{g} in G and a subsequence $\{g_{i_j}: j \in \mathbb{N}\}$ of $\{g_i: i \in \mathbb{N}\}$, such that

$$g_{i_j} \notin W, \quad j \in \mathbb{N}.$$

By (8) we have that

$$g_{i_j} \in A = \check{g}\overline{G[U]} - W, \quad (9)$$

for all $j \in \mathbb{N}$. Since A is compact the sequence $\{g_{i_j}\}_{j \in \mathbb{N}}$, considered as a net, has a subnet $\{g_\lambda\}_{\lambda \in \Lambda}$ that converges to a point \bar{g} in A , i.e.,

$$\lim_{\lambda \in \Lambda} g_\lambda = \bar{g} \in A. \quad (10)$$

Now $\hat{g} \in W$ and $\bar{g} \in A \subset G - W$, and hence

$$\hat{g} \neq \bar{g}. \quad (11)$$

Since the action of G on X is continuous, and since (5), (6) and (10) hold, we obtain

$$\bar{x} = \lim_{\lambda \in \Lambda} x_\lambda = \lim_{\lambda \in \Lambda} g_\lambda x'_\lambda = \left(\lim_{\lambda \in \Lambda} g_\lambda \right) \left(\lim_{\lambda \in \Lambda} x'_\lambda \right) = \bar{g} \bar{x}'. \quad (12)$$

Since the action of G on X is free it follows by (4) and (12) that $\hat{g} = \bar{g}$. This contradicts (11), and hence (3) holds, which completes the proof. \square

Theorem 5.3. Let G be a topological group and let X be a free Cartan G -space, such that the natural projection $\pi: X \rightarrow X/G$ is a Serre fibration. Suppose k is a coefficient system for G , over a ring R , with the property that each G -map $\alpha: G \rightarrow G$ induces $\alpha_* = \text{id}: k(G) \rightarrow k(G)$. Then there exists a natural isomorphism

$$\mathcal{E}: H_n^G(X; k) \xrightarrow{\cong} H_n(X/G; k(G)), \quad \text{for each } n \geq 0.$$

The result also holds in the relative case, for a G -pair (X, A) .

Proof. As the case was in Theorem 4.1, there exists an isomorphism already on the level of chain complexes. Thus we shall construct a chain map

$$\xi: S^G(X; k) \rightarrow S(X/G; k(G)), \quad (1)$$

and show that it is an isomorphism. First we construct a homomorphism

$$\hat{\xi} : \hat{S}_n^G(X; k) \rightarrow S_n(X/G; k(G)), \quad \text{for each } n \geq 0, \quad (2)$$

as follows. Since the action of G on X is free every equivariant singular n -simplex in X is of type $\{e\}$. Let $T : \Delta_n \times G \rightarrow X$ be such an equivariant singular n -simplex in X , and let $a \in k(G)$. Then we define

$$\hat{\xi}(T \otimes a) = T^* \otimes a,$$

and extend linearly. Here $T^* : \Delta_n \rightarrow X/G$ denotes the map induced by T in the orbit spaces, see (2) in Section 1.3. It is clear from the definition of $\hat{\xi}$ that $\partial \circ \hat{\xi} = \hat{\xi} \circ \partial$, and hence the homomorphisms $\hat{\xi}$ in (2), for all $n \geq 0$, form a chain map $\hat{\xi} : \hat{S}_n^G(X; k) \rightarrow S_n(X/G; k(G))$.

If $T \otimes a \sim T' \otimes b \in \hat{S}_n^G(X; k)$ there exists a G -map $\eta : \Delta_n \times G \rightarrow \Delta_n \times G$, which covers id_{Δ_n} , such that $T = T' \circ \eta$ and $\eta_*(a) = b$, where $\eta_* : k(G) \rightarrow k(G)$. Since $\eta_* = \text{id}$, we have that $a = b$. Also $T^* = (T')^*$, and hence $\hat{\xi}(T \otimes a) = T^* \otimes a = (T')^* \otimes b = \hat{\xi}(T' \otimes b)$. Therefore $\hat{\xi}$ in (2) induces a homomorphism

$$\xi : S_n^G(X; k) \rightarrow S_n(X/G; k(G)), \quad \text{for each } n \geq 0, \quad (3)$$

and thus we obtain a chain map ξ as in (1).

Next we construct a homomorphism

$$\sigma : S_n(X/G; k(G)) \rightarrow S_n^G(X; k), \quad \text{for each } n \geq 0. \quad (4)$$

Let $D : \Delta_n \rightarrow X/G$ be a singular n -simplex in X/G . Since $\pi : X \rightarrow X/G$ is a Serre fibration there exists a lifting $F : \Delta_n \rightarrow X$ of D to X , i.e., $\pi \circ F = D$. We then define, for each $a \in k(G)$,

$$\sigma(D \otimes a) = [\langle F \rangle \otimes a]. \quad (5)$$

Here $\langle F \rangle : \Delta_n \times G \rightarrow X$, $(v, g) \mapsto gF(v)$, is an equivariant singular n -simplex, of type $\{e\}$, in X , see (5) in Section 1.3. Thus $\langle F \rangle \otimes a \in \hat{S}_n^G(X; k)$, and $[\langle F \rangle \otimes a] = p([\langle F \rangle \otimes a]) \in S_n^G(X; k)$.

We need to verify that the definition in (5) is well defined, i.e., that it is independent of the choice of the lifting F of D . Suppose $F, F' : \Delta_n \rightarrow X$ are two liftings of $D : \Delta_n \rightarrow X/G$. By Lemma 5.2 there then exists a continuous function $\kappa : \Delta_n \rightarrow G$ such that $F(v) = \kappa(v)F'(v)$, for all $v \in \Delta_n$. We now obtain a commutative diagram

$$\begin{array}{ccc} \Delta_n \times G & & \\ \eta \downarrow & \searrow \langle F \rangle & \\ \Delta_n \times G & \nearrow \langle F' \rangle & X \end{array}$$

where η is defined by $\eta(v, g) = (v, g\kappa(v))$, for all $(v, g) \in \Delta_n \times G$. Then η is a G -map (in fact a G -homeomorphism), which covers id_{Δ_n} , and hence

$$\langle F \rangle \otimes a \sim \langle F' \rangle \otimes \eta_*(a) = \langle F' \rangle \otimes a,$$

since $\eta_* = \text{id}$. Thus $[\langle F \rangle \otimes a] = [\langle F' \rangle \otimes a]$, and this shows that the definition in (5) is independent of the choice of the lifting F of D .

We shall now show that $\sigma \circ \xi = \text{id}$. Let $T : \Delta_n \times G \rightarrow X$ be an equivariant singular n -simplex in X , and let $a \in k(G)$. Then $T_e : \Delta_n \rightarrow X$, $v \mapsto T(v, e)$, is a lifting of $T^* : \Delta_n \rightarrow X/G$ to X . Thus

$$(\sigma \circ \xi)([T \otimes a]) = \sigma(T^* \otimes a) = [\langle T_e \rangle \otimes a] = [T \otimes a],$$

since $\langle T_e \rangle = T : \Delta_n \times G \rightarrow X$, see (7) in Section 1.3.

Next we show that $\xi \circ \sigma = \text{id}$. Let $D : \Delta_n \rightarrow X/G$ be a singular n -simplex in X/G , and let $a \in k(G)$. Choose a lifting $F : \Delta_n \rightarrow X$ of D to X . Then

$$(\xi \circ \sigma)(D \otimes a) = \xi([\langle F \rangle \otimes a]) = \langle F \rangle^* \otimes a = D \otimes a$$

since $\langle F \rangle^* = D$, see (5) in Section 1.3. Thus we have showed that ξ in (3) is an isomorphism, for each $n \geq 0$, and hence the chain map ξ in (1) is an isomorphism of chain complexes. \square

The proof of Theorem 5.5 relies on Theorem 5.3 and Lemma 5.4 below. In the case where G is a p -adic group A_p , p a prime, the result in Lemma 5.4 is attributed to C.N. Lee in [11, Section 3].

Lemma 5.4. Suppose X is a free G -space, where G is a totally disconnected compact group. Then the natural projection $\pi : X \rightarrow X/G$ is a fibration, i.e., it has the homotopy lifting property. Moreover $\pi : X \rightarrow X/G$ has the unique path lifting property.

Proof. By Theorem Π in the introduction we know that G is a profinite group. Thus

$$G = \varprojlim_{\alpha \in \Lambda} S_{\alpha}, \quad (1)$$

where $\Lambda = (\Lambda, <)$ is a directed set and $\{S_{\alpha}\}_{\alpha \in \Lambda} = \{S_{\alpha}, \varphi_{\alpha}^{\beta}\}_{\alpha \in \Lambda}$ is an inverse system of finite groups S_{α} , $\alpha \in \Lambda$, and homomorphisms $\varphi_{\alpha}^{\beta} : S_{\beta} \rightarrow S_{\alpha}$, for each pair $\alpha, \beta \in \Lambda$, with $\alpha < \beta$. Thus

$$G = \left\{ \{s_{\alpha}\}_{\alpha \in \Lambda} \in \prod_{\alpha \in \Lambda} S_{\alpha} \mid s_{\alpha} = \varphi_{\alpha}^{\beta}(s_{\beta}), \text{ if } \alpha < \beta, \alpha, \beta \in \Lambda \right\}. \quad (2)$$

For each fixed $\gamma \in \Lambda$ we define

$$H_{\gamma} = \{ \{s_{\alpha}\}_{\alpha \in \Lambda} \in G \mid s_{\gamma} = e \}. \quad (3)$$

The map

$$p_{\gamma} : G \rightarrow S_{\gamma}, \quad \{s_{\alpha}\}_{\alpha \in \Lambda} \mapsto s_{\gamma}, \quad (4)$$

is a continuous homomorphism, and

$$\text{Ker } p_{\gamma} = H_{\gamma}. \quad (5)$$

Thus each H_{γ} , $\gamma \in \Lambda$, is an open and closed, normal subgroup of G . In particular each H_{γ} , $\gamma \in \Lambda$, is compact. The homomorphism p_{γ} in (4) induces an injective homomorphism

$$\bar{p}_{\gamma} : G/H_{\gamma} \rightarrow S_{\gamma}, \quad (6)$$

and hence G/H_{γ} is a finite group for each $\gamma \in \Lambda$.

Furthermore we note that since $G = \varprojlim_{\alpha \in \Lambda} S_{\alpha}$ has the relative topology from the product $\prod_{\alpha \in \Lambda} S_{\alpha}$ with the product topology, it follows that if U is any open neighborhood of $e \in G$, then there exists $\gamma \in \Lambda$ such that

$$H_{\gamma} \subset U. \quad (7)$$

For each $\alpha \in \Lambda$ the natural projection $\pi : X \rightarrow X/G$ factors through X/H_{α} , i.e., we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\pi_{\alpha}} & X/H_{\alpha} \\ & \searrow \pi & \downarrow q^{\alpha} \\ & & X/G \end{array} \quad (8)$$

Here π_{α} denotes the projection onto the orbit space X/H_{α} of the induced action of H_{α} on X . Since H_{α} is compact it follows that π_{α} is a closed map and that X/H_{α} is Hausdorff, see e.g. [1, Theorem I.3.1].

Since H_{α} is a closed normal subgroup of G there is an induced action $G/H_{\alpha} \times X/H_{\alpha} \rightarrow X/H_{\alpha}$, $(gH_{\alpha}, \pi_{\alpha}(x)) \mapsto \pi_{\alpha}(gx)$, of the quotient group G/H_{α} on X/H_{α} . The orbit space of this action equals $(X/H_{\alpha})/(G/H_{\alpha}) = X/G$, and q^{α} in (8) denotes the projection onto the orbit space. Since the action of G on X is free it follows that the induced action of G/H_{α} on X/H_{α} is free. As we already noted above, see (6), G/H_{α} is a finite group, and thus q^{α} , in (8), is a covering projection, and hence a fibration with unique path lifting, see e.g. [13, Theorems 2.2.3 and 2.2.5].

Now suppose that $\alpha < \beta$, where $\alpha, \beta \in \Lambda$. Then $H_{\beta} \subset H_{\alpha}$, and hence we have a natural projection map

$$\pi_{\alpha}^{\beta} : X/H_{\beta} \rightarrow X/H_{\alpha}. \quad (9)$$

Furthermore we have that

$$\pi_{\alpha} = \pi_{\alpha}^{\beta} \circ \pi_{\beta}, \quad \text{whenever } \alpha < \beta. \quad (10)$$

Here π_{α} and π_{β} are as in (8). The spaces X/H_{α} , $\alpha \in \Lambda$, and the maps $\pi_{\alpha}^{\beta} : X/H_{\beta} \rightarrow X/H_{\alpha}$, for each pair $\alpha, \beta \in \Lambda$ with $\alpha < \beta$, form an inverse system $\{X/H_{\alpha}, \pi_{\alpha}^{\beta}\}_{\alpha \in \Lambda}$. We claim that the map

$$\pi_{\infty} : X \rightarrow \varprojlim_{\alpha \in \Lambda} X/H_{\alpha}, \quad x \mapsto \{ \pi_{\alpha}(x) \}_{\alpha \in \Lambda}, \quad (11)$$

is a homeomorphism. It follows by (10) that π_∞ is well defined, and clearly π_∞ is continuous. Next we show that π_∞ is injective. Let $x_1, x_2 \in X$, where $x_1 \neq x_2$. Since the action of G on X is continuous, there exists an open neighborhood U of $e \in G$, such that $Ux_1 \subset X \setminus \{x_2\}$. Now let $\gamma \in \Lambda$ be such that $H_\gamma \subset U$, see (7). Then $H_\gamma x_1 \subset Ux_1 \subset X \setminus \{x_2\}$. Thus $x_2 \notin H_\gamma x_1$, and hence $H_\gamma x_1 \cap H_\gamma x_2 = \emptyset$. This means that $\pi_\gamma(x_1) \neq \pi_\gamma(x_2) \in X/H_\gamma$, and hence $\pi_\infty(x_1) \neq \pi_\infty(x_2)$. Thus π_∞ in (11) is injective.

The fact that π_∞ is surjective is seen as follows. Let $y = \{y_\alpha\}_\alpha \in \varprojlim_{\alpha \in \Lambda} (X/H_\alpha)$, where $y_\alpha \in X/H_\alpha$, for every $\alpha \in \Lambda$, and $y_\alpha = \pi_\alpha^\beta(y_\beta)$, whenever $\alpha < \beta$, $\alpha, \beta \in \Lambda$. Let us denote

$$X_\alpha = \pi_\alpha^{-1}(y_\alpha).$$

Then X_α is a compact subset of X , for each $\alpha \in \Lambda$, and

$$X_\beta \subset X_\alpha, \quad \text{if } \alpha < \beta. \quad (12)$$

Let $\alpha_0 \in \Lambda$ be fixed, and denote $X_{\alpha_0} = X_0$. Then $\{X_0 \cap X_\alpha\}_{\alpha \in \Lambda}$ is a family of closed subsets of the compact space X_0 , and since $\Lambda = (\Lambda, <)$ is a directed set and (12) holds it has the finite intersection property. Thus $\bigcap_{\alpha \in \Lambda} X_\alpha = \bigcap_{\alpha \in \Lambda} (X_0 \cap X_\alpha) \neq \emptyset$. Let $x \in \bigcap_{\alpha \in \Lambda} X_\alpha$. Then $\pi_\alpha(x) = y_\alpha$, for every $\alpha \in \Lambda$, and hence $\pi_\infty(x) = \{y_\alpha\}_{\alpha \in \Lambda} = y$. Thus π_∞ is surjective.

Suppose B is a closed subset of X . Then the set $\prod_{\alpha \in \Lambda} \pi_\alpha(B)$ is closed in $\prod_{\alpha \in \Lambda} (X/H_\alpha)$, since, as we noted in (8), each $\pi_\alpha : X \rightarrow X/H_\alpha$, $\alpha \in \Lambda$, is a closed map. Hence

$$\pi_\infty(B) = \prod_{\alpha \in \Lambda} \pi_\alpha(B) \cap \varprojlim_{\alpha \in \Lambda} (X/H_\alpha)$$

is closed in $\varprojlim_{\alpha \in \Lambda} (X/H_\alpha)$, and thus π_∞ is a closed map. We have now shown that π_∞ is a closed continuous map, and a bijection, and hence π_∞ is a homeomorphism.

Note also that we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\pi_\infty} & \varprojlim_{\alpha \in \Lambda} (X/H_\alpha) \\ & \searrow \pi & \swarrow q \\ & X/G & \end{array}$$

where q is defined by $\{y_\alpha\}_{\alpha \in \Lambda} \mapsto q_\alpha(y_\alpha)$, and this definition is well defined since $q^\beta = q^\alpha \circ \pi_\alpha^\beta$, for each $\alpha < \beta$, where $\alpha, \beta \in \Lambda$.

Now suppose Z is any topological space, and let

$$F : Z \times I \rightarrow X/G \quad (13)$$

be a homotopy which starts at $f : Z \rightarrow X/G$, and suppose that $\tilde{f} : Z \rightarrow X$ is a lifting of f to X . Our claim is that there exists a homotopy

$$\tilde{F} : Z \times I \rightarrow X, \quad (14)$$

which starts at \tilde{f} , and which is a lifting of F to X .

As we noted in (8), each projection $q^\alpha : X/H_\alpha \rightarrow X/G$, $\alpha \in \Lambda$, is a fibration with unique path lifting. Hence there exists, for each $\alpha \in \Lambda$, a unique homotopy

$$\tilde{F}_\alpha : Z \times I \rightarrow X/H_\alpha,$$

which starts at

$$\tilde{f}_\alpha = \pi_\alpha \circ \tilde{f} : Z \rightarrow X/H_\alpha$$

and which lifts F to X/H_α . It follows by the uniqueness property that if $\alpha < \beta$, where $\alpha, \beta \in \Lambda$, then $\tilde{F}_\alpha = \pi_\alpha^\beta \circ \tilde{F}_\beta$. Thus we obtain a well-defined map

$$\hat{F} : Z \times I \rightarrow \varprojlim_{\alpha \in \Lambda} (X/H_\alpha), \quad (z, t) \mapsto \{\tilde{F}_\alpha(z, t)\}_{\alpha \in \Lambda}.$$

Now

$$\tilde{F} = (\pi_\infty)^{-1} \circ \hat{F} : Z \times I \rightarrow X$$

is a homotopy, which starts at $\tilde{f} : Z \rightarrow X$, and which lifts F to X . \square

Theorem 5.5. Let X be a free G -space, where G is a totally disconnected compact group, and let A be a G -invariant subspace of X . Suppose k is a coefficient system for G , over a ring R , with the property that each G -map $\alpha : G \rightarrow G$ induces $\alpha_* = \text{id} : k(G) \rightarrow k(G)$. Then there exists a natural isomorphism

$$\mathcal{E} : H_n^G(X; k) \xrightarrow{\cong} H_n(X/G; k(G)), \quad \text{for each } n \geq 0.$$

The result also holds in the relative case for a free G -pair (X, A) , in place of the G -space X .

Proof. Note that since G is compact the action of G on X is Cartan. Theorem 5.5 now follows by Lemma 5.4 and Theorem 5.3. \square

6. Two exact homology sequences

Suppose G is a totally disconnected locally compact group, and let R be a ring. By \underline{A} we denote the natural coefficient system for G , over R , see Section 4. We let \underline{R} denote the (canonical) constant coefficient system for G , over the ring R . That is

$$\underline{R}(G/H) = R, \quad \text{for every } H < G,$$

and for each G -map $\alpha : G/H \rightarrow G/K$, where $H, K < G$, the induced homomorphism $\alpha_* : \underline{R}(G/H) \rightarrow \underline{R}(G/K)$ is the identity map on R , i.e.,

$$\alpha_* = \text{id}_R : R \rightarrow R.$$

Next we define a natural transformation of coefficient systems

$$\mu : \underline{A} \rightarrow \underline{R}$$

as follows. For each $H < G$ we define $\mu : \underline{A}(G/H) \rightarrow \underline{R}(G/H)$ to be the homomorphism

$$\mu : R[G/H] \rightarrow R, \quad \sum_{i=1}^m r_i(g_i H) \mapsto \sum_{i=1}^m r_i.$$

We call μ the *augmentation homomorphism*. Then we define the *augmentation coefficient system* I by

$$I = \text{Ker}(\mu : \underline{A} \rightarrow \underline{R}).$$

That is, for each $H < G$, we set $I(G/H) = \text{Ker}(\mu : R[G/H] \rightarrow R)$. If $\alpha : G/H \rightarrow G/K$ is a G -map, then $\alpha_* : I(G/H) \rightarrow I(G/K)$ is the restriction of $\alpha_* : R[G/H] \rightarrow R[G/K]$ to $I(G/H)$. Clearly I is a covariant coefficient system for G , over R , and we obtain a short exact sequence of coefficient systems

$$0 \rightarrow I \xrightarrow{\iota} \underline{A} \xrightarrow{\mu} \underline{R} \rightarrow 0. \quad (1)$$

By Theorem 3.1 this gives us, for any G -space X , an induced long exact sequence of equivariant singular homology groups

$$\cdots \xrightarrow{\Delta} H_n^G(X; I) \xrightarrow{\iota_*} H_n^G(X; \underline{A}) \xrightarrow{\mu_*} H_n^G(X; \underline{R}) \xrightarrow{\Delta} H_{n-1}^G(X; I) \xrightarrow{\iota_*} \cdots. \quad (2)$$

Since Theorem 3.1 also holds in the relative case there is a corresponding exact sequence, for any G -pair (X, A) in place of X . Using Theorem 4.1 we now obtain.

Theorem 6.1. Suppose X is a G -space, where G is a totally disconnected locally compact group, and let the coefficient systems \underline{R} and I be as above. Then there exists a long exact sequence

$$\cdots \xrightarrow{\Delta} H_n^G(X; I) \xrightarrow{i} H_n(X; R) \xrightarrow{\bar{\mu}} H_n^G(X; \underline{R}) \xrightarrow{\Delta} H_{n-1}^G(X; I) \xrightarrow{i} \cdots.$$

Proof. By Theorem 4.1 there exists an isomorphism $\Phi : H_n^G(X; \underline{A}) \rightarrow H_n(X; R)$, for each $n \geq 0$. We set $i = \Phi \circ \iota_*$ and $\bar{\mu} = \mu_* \circ \Phi^{-1}$, where ι_* and μ_* are as in (2). The claim now follows by the exactness of the sequence (2) above. \square

For free actions of totally disconnected compact groups, that is, for free actions of profinite groups, we obtain Theorem 6.3 below. The proof of Theorem 6.3 uses Theorems 4.1 and 5.5 together with Theorem 3.1, more precisely, together with the long exact sequence (2) above. First we need the following lemma.

Lemma 6.2. Let X be a G -space, where G is a totally disconnected compact group. Then

$$\mathcal{E} \circ \mu_* \circ \Phi^{-1} = \pi_* : H_n(X; R) \rightarrow H_n(X/G; R),$$

where $\pi : X \rightarrow X/G$ denotes the natural projection.

Proof. It is enough to show that on the level of chain complexes we have that

$$\pi_{\#} = \xi \circ \mu_{\#} \circ \psi : S_n(X; R) \rightarrow S_n^G(X, \Lambda) \rightarrow S_n^G(X; \underline{R}) \rightarrow S_n(X/G; R).$$

Here ψ is as in (9) in the proof of Theorem 4.1, $\mu_{\#}$ is as in (7) in Section 3 and ξ is as in (3) in the proof of Theorem 5.3. Let $F : \Delta_n \rightarrow X$ be an ordinary singular n -simplex in X , and let $1 \in R$. Then

$$(\xi \circ \mu_{\#} \circ \psi)(F \otimes 1) = (\xi \circ \mu_{\#})([(F) \otimes e]) = \xi([(F) \otimes 1]) = \langle F \rangle^* \otimes 1 = (\pi \circ F) \otimes 1 = \pi_{\#}(F \otimes 1),$$

and hence our claim follows. \square

Theorem 6.3. Suppose X is a free G -space, where G is a totally disconnected compact group. Let I denote the augmentation coefficient system for G , over a ring R . Then there is a long exact sequence

$$\xrightarrow{\bar{\Delta}} H_n^G(X; I) \xrightarrow{i} H_n(X; R) \xrightarrow{\pi_*} H_n(X/G; R) \xrightarrow{\bar{\Delta}} H_{n-1}^G(X; I) \xrightarrow{i} \dots$$

Proof. By Theorem 4.1 there exists an isomorphism $\Phi : H_n^G(X; \Lambda) \rightarrow H_n(X; R)$, for every $n \geq 0$. By Theorem 5.5 there exists an isomorphism $\mathcal{E} : H_n^G(X; \underline{R}) \rightarrow H_n(X/G; R)$, for every $n \geq 0$. We define $\bar{i} = \Phi \circ \iota_*$ and $\bar{\Delta} = \Delta \circ \mathcal{E}^{-1}$, where ι_* and Δ are as in (2). By Lemma 6.2 we have that $\mathcal{E} \circ \mu_* \circ \Phi^{-1} = \pi_* : H_n(X; R) \rightarrow H_n(X/G; R)$, for every $n \geq 0$, where $\pi : X \rightarrow X/G$ denotes the natural projection onto the orbit space. Our claim now follows by the exactness of the sequence (2) above. \square

Appendix A. Homogeneous spaces of totally disconnected locally compact groups

We used Proposition A.1 below in the proof of Lemma 2.1. Proposition A.1 is of course a well-known result, but in order to be complete we record a proof here.

Proposition A.1. Let G be a totally disconnected locally compact group, and let $H < G$. Then G/H is totally disconnected.

For the lemma and corollary below, see e.g. [9, the second lemma and the corollary on p. 38].

Lemma A.2. Let X be a compact Hausdorff space, and let $x \in X$. Let $\{A_{\alpha}\}_{\alpha \in \Lambda}$ be the family of all subsets of X that are both open and closed in X and that contain x . Then the set $C = \bigcap_{\alpha \in \Lambda} A_{\alpha}$ equals the connected component of X that contains x .

Corollary A.3. Let C be a connected component of a compact Hausdorff space X , and let U be any open neighborhood of C in X . Then there exists an open and closed neighborhood B of C in X , such that $B \subset U$.

One now obtains in a standard way the following version of Corollary A.3 for locally compact Hausdorff spaces.

Corollary A.4. Let C be a compact connected component of a locally compact Hausdorff space X , and let U be any open neighborhood of C in X . Then there exists an open and compact neighborhood B of C in X , such that $B \subset U$.

Proof of Proposition A.1. Let $p : G \rightarrow G/H$ denote the natural projection. Suppose L is a connected subset of G/H , and that $gH, \bar{g}H \in L$, where $gH \neq \bar{g}H$. By Corollary A.4 there exists an open and compact neighborhood B of the point g in G , such that $B \cap \bar{g}H = \emptyset$. Since $p(B)$ is compact, and G/H is Hausdorff, see e.g. [9, Theorem on p. 27], it follows that $p(B)$ is closed in G/H . Thus $p(B)$ is an open and closed subset of G/H . Now $gH \in p(B) \cap L$, and hence $p(B) \cap L$ is an open and closed, non-empty subset of L . Therefore $p(B) \cap L = L$, which contradicts the fact that $\bar{g}H \in L$ and $\bar{g}H \notin p(B)$. Thus L consists of a single point, which shows that G/H is totally disconnected. \square

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